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THE GENERALIZED INVERSE PROBLEM OF ORBIT COMPUTATION *

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Abstract: The basic problem of celestial mechanics — finding the motion for given initial conditions in a given force field — is inverted by asking for force fields which result in either a specific orbit or which allow to obtain a closed form solution of the equations of motion. The first case leads to a generalized concept of guidance and thrust programs, while the second opens the way to finding new closed form solutions for slightly modified n-body ($n > 2$) gravitational fields.

The problem discussed in this paper is a generalization in purpose and in method of the classical problem of orbit determination in celestial mechanics. It allows the determination of classes of functions which either furnish optimization opportunities of the guidance problem (first case) or furnish overall field modifications which allow closed form solutions (second case) applicable to perturbation calculations as reference orbits with predetermined approximations.

Establishing new reference orbits for perturbations and formulating a concise and systematic method of guidance analysis are the two main results presented in the paper.

Резюме: Описывается метод, с помощью котерого определяются модификации сил, действующих на космические ракеты таким образом, что их орбиты и траектории могут быть описаны и представлены с помощью решений в замкнутой форме с данным приближением. Основная задача небесной механики – нахождение движения для данных начальных условий в данном силовом поле – обращается и сводится к задаче нахождения силового поля, которое дает в результате или определенную орбиту, или позволяет получить решение в замкнутой форме уравнений движения. В первом случае данное силовое поле модифицируется вдоль траектории, вторая задача требует модификации полного поля. Первый случай ведет к обобщенной концепции программ управления и тяги, тогда как второй открывает пути к нахождению новых решений в замкнутой форме для слабо модифицированных гравитационных полей n тела ($n > 2$). Классическая задача определения орбиты в небесной механике состоит в нахождении элементов орбиты и (или) начальных условий на основе наблюдений в силовом поле, т.е. представляет собой задачу обратную задаче расчета траектории для данного силового поля и данных начальных условий. Она может быть модифицирована с целью нахождения определенных констант или характеристик силового поля, закон тяготения которого предполагается заданным (например, поле Земли или открытие Нептуна). Рассматриваемая в статье обратная задача представляет собой обобщение по цели и методу, так как она позволяет определить классы функций, которые обеспечивают либо оптимальные возможности проблемы управления (первый случай), либо общие модификации поля, позволяющие получить решения в замкнутой форме (второй случай), применимые к расчетам с возмущениями исходных орбит с заранее определенными приближениями. Установление новых исходных орбит при наличии возмущений и формулирование точного и систематического метода анализа наведения – два основных результата настоящей работы.

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1. Introduction

Inverse techniques are widely used in several branches of theoretical physics, applied mathematics, continuum mechanics, celestial mechanics, etc. In some of these fields the inverse approach has lead to the solution of important problems, in others it opened up entirely new fields of research and in a few cases the potentialities have not yet been fully evaluated. The more flexible the concept of inverse technique is, the more usefulness it offers, therefore, in the following, a general view will be presented and applied to some basic problems of applied celestial mechanics.

A phenomenon is often described by a set of differential equations and initial and boundary conditions. A "solution" is often presented by a set of independent variables as functions of space, time or other dependent variables. These functions are considered solutions if they satisfy the above mentioned equations and conditions. The idea of the inverse approach stated in its most general form is to start out with the "solution" and find the equations describing the phenomenon. Inasmuch as this general statement is of very limited use for solving actual problems, certain restrictions are applied. In hydrodynamics for instance, the continuity equation for incompressible potential flow is the Laplace equation which the potential function (φ) must satisfy. Therefore, functions which satisfy the $\Delta\varphi = 0$ equation will describe certain flow fields. The real and imaginary parts of functions of a complex variable satisfy the Laplace equation, therefore, flow fields might be generated *inversely* by finding the boundary conditions which are satisfied by arbitrary chosen functions. To satisfy part of the problem by giving solutions of the differential equations and then finding the boundary conditions is a well known approach in continuum mechanics.

On the other hand, to find the equation which will be satisfied by a "solution" which in turn satisfies certain initial conditions is a frequently encountered problem in celestial mechanics. Even the above, already restricted — as compared to the original — statement is too general, since often only certain constants in the differential equations are found by imposing a solution. Frequently the general form of the equations are given and the "solution" will be selected so that it will further specify the equations. The classical example is, of course, when observational data represent the "solution" and the equations describing the phenomenon are sought. Some of the physical laws governing the phenomenon are known and some are unknown, some numerical constants attached to the problem are given, some are to be determined. Classical examples are the discovery of Neptune from its perturbative effect on the orbit of Uranus, establishment of the coefficients of the

higher order harmonics in the expansion of the Earth's gravitational potential, etc.

Another example of somewhat different type is the solution of the earth satellite problem by Vinti [1] and Garfinkel [2]. Slight modifications of the potential function describing the Earth's gravitational field either by adjusting numerical coefficients or by changing the functions – might result in satisfying the new equations by a "solution" which would not satisfy the original equations.

Several questions remain open in connection with the above mentioned examples. Differential equations describe physical phenomena only approximately and it can happen that a certain set of data will "fit" a solution of the modified equations better than it satisfies the original equations. If the numerical constants occurring in an equation are modified and this way a solution is obtained, the deviation must be evaluated between the solutions to the original and to the modified equations. If not only numerical constants but the functions involved are modified, this evaluation might be very difficult.

The avenues of applications of the inverse approach in celestial mechanics have been varied and proven to be extremely useful. The purpose of this paper is to present a general view and to point out its applications.

2. Analytical considerations

2.1. Elimination of time

Consider dynamical systems of n degrees of freedom with generalized coordinates $q_1, q_2, \ldots, q_i, \ldots, q_n$, and write the equations of motion in the form:

$$\ddot{q}_i = f_i(q_k, \dot{q}_k), \quad i, k = 1(1)n. \tag{1}$$

Considering every q_i depending on q_1, we have

$$q_i = q_i(q_1) \quad \text{and} \quad \dot{q}_i = \frac{dq_i}{dq_1}\dot{q}_1$$

or

$$\dot{q}_i = q_i'\dot{q}_1 \tag{2}$$

where the prime denotes derivatives with respect to q_1 and the dot denotes derivatives with respect to the time (t). From (2):

$$\ddot{q}_i = q_i'\ddot{q}_1 + q_i''(\dot{q}_1)^2 \tag{3}$$

For $i = 2$, eqs. (1) and (3) give

$$f_2(q_k, \dot{q}_k) = q_2' f_1(q_k, \dot{q}_k) + q_2''(\dot{q}_1)^2$$

or using eq. (2):

$$f_2(q_k, q_k', \dot{q}_1) = q_2' f_1(q_k, q_k', \dot{q}_1) + q_2''(\dot{q}_1)^2. \tag{4}$$

Solving this equation for $\dot{q}_1$, or for $(\dot{q}_1)^2$, we write

$$(\dot{q}_1)^2 = H(q_k, q_k', q_2'') \tag{5}$$

and computing the time derivative, we obtain

$$2\dot{q}_1\ddot{q}_1 = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k}\dot{q}_k + \sum_{k=2}^{n} \frac{\partial H}{\partial q_k'}\frac{dq_k'}{dt} + \frac{\partial H}{\partial q_2''}\frac{dq_2''}{dt}. \tag{6}$$

Eqs. (1), (2), and (5) transform (6) into

$$2f_1(q_k, q_k', q_2'') = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} q_k' + \sum_{k=2}^{n} \frac{\partial H}{\partial q_k'} q_k'' + \frac{\partial H}{\partial q_2''} q_2''' \tag{7}$$

where $q_1' \equiv 1$, provided that $\dot{q}_1 \neq 0$.

For $i=3$, eq. (3) gives

$$f_3(q_k, q_k', q_2'') = q_3' f_1(q_k, q_k', q_2'') + q_3'' H(q_k, q_k', q_2''). \tag{8}$$

The form of the equation is the same for $i=3(1)n$:

$$f_i(q_k, q_k', q_2'') = q_i' f_1(q_k, q_k', q_2'') + q_i'' H(q_k, q_k', q_2''). \tag{9}$$

Eq. (7) is a third order differential equation and the $(n-2)$ eqs. (9) are of the second order. The system is of $(2n-1)$ -th order as expected since it was obtained by elimination of the time from the original ($2n$-th order) set. It is significant that eq. (7) contains only one third derivative. In the two degrees of freedom case $i=1, 2$ and only eq. (7) is obtained by elimination of the time. This equation, however, will describe completely the geometry of the problem.

Let $\quad q_1 = x,\ q_2 = y, \quad$ and

$$\left.\begin{aligned} \ddot{x} &= f_1(x, y, \dot{x}, \dot{y}) \\ \ddot{y} &= f_2(x, y, \dot{x}, \dot{y}) \end{aligned}\right\} \tag{10}$$

corresponding to eqs. (1). The simple relations of $\dot{y} = y'\dot{x}$ with $y' = dy/dx$ and $y = y(x)$, correspond to eqs. (2). Eq. (4) becomes:

$$f_2(x, y, y', \dot{x}) = y' f_1(x, y, y', \dot{x}) + y''(\dot{x})^2 \tag{11}$$

which, when solved for $\dot{x}$ for $(\dot{x})^2$ gives:

$$(\dot{x})^2 = H(x, y, y', y''). \tag{12}$$

Final elimination of the time results in the equation of the orbit, corres-

ponding to eq. (7):

$$2f_1(x, y, y', y'') = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''}\right) H(x, y, y', y''). \qquad (13)$$

The above equations of motion (10) represent a slightly more general case than the two dimensional restricted three body problem in a rotating rectangular coordinate system. The differential equation connecting y and x is of the third order. If use is made of the Jacobi integral, eq. (13) can be reduced to the second order, as will be shown later. The three dimensional case of the restricted three body problem corresponds to

$$\left.\begin{aligned} \ddot{x} &= f_1(x, y, \dot{y}) \\ \ddot{y} &= f_2(x, y, \dot{x}) \\ \ddot{z} &= f_3(x, y) \end{aligned}\right\} \qquad (14)$$

and it can be shown that the 6-th order system is reduced to the 5-th order, consisting of a third order and of a second order differential equation. The first equation contains

$$x, y, z, y', z', y'', z'', y'''$$

and the second

$$x, y, z, y', z', y'', z''.$$

The following three remarks are offered at this point:

(i) By eliminating the time two purposes were served: (a) the order of the system of differential equations has been reduced by one, and (b) since time effects do not enter, the problem is changed into a geometrical one.

(ii) If the equations of motion contain the time explicitly, eq. (5) will contain an additional term $\partial H/\partial t$, in which case elimination of time is not achieved.

(iii) Since the elimination of time process implies that the $q_1 = q_1(t)$ function be inverted to $t = t(q_1)$, and substituted in the $q_i = q_i(t)$ relation, giving $q_i = q_i[t(q_1)]$ the $\dot{q}_1 = 0$ point requests special attention. Double valued functions occurring in eq. (5) warrant the same comment.

2.2 Modified equations

Consider the right side of eqs. (1) as the sum of two functions:

$$f_i(q_k, \dot{q}_k) = F_i(q_k, \dot{q}_k) + \varphi_i(q_k) \qquad (15)$$
$$k, i = 1(1)n$$

i.e the modifying functions (φ_i) depend only on the generalized position coordinates. The effect of the φ_i set will now be investigated on the geometry of the orbit.

Eq. (4) will assume the form of

$$F_2(q_k, q_k', \dot{q}_1)+\varphi_2(q_k)=q_2'[F_1(q_k, q_k', \dot{q}_1)+\varphi_1(q_k)]+q_2''(\dot{q}_1)^2. \tag{16}$$

From this equation we obtain $\dot{q}_1$, assuming that $F_1(q_k, q_k', \dot{q}_1)$ and $F_2(q_k, q_k', \dot{q}_1)$ are given and the $\varphi_1(q_k)$ and $\varphi_2(q_k)$ functions are not.

$$\dot{q}_1^2=H(q_k, q_k', q_2'', \varphi_1, \varphi_2). \tag{17}$$

The first equation of the orbit corresponding to eq. (7) is obtained again by differentiation:

$$\left.\begin{aligned} 2F_1+2\varphi_1= & \sum_{k=1}^{n}\frac{\partial H}{\partial q_k}q_k'+\sum_{k=2}^{n}\frac{\partial H}{\partial q_k'}q_k''+\frac{\partial H}{\partial q_2''}q_2'''+\frac{\partial H}{\partial \varphi_1}\sum_{k=1}^{n}\frac{\partial \varphi_1}{\partial q_k}q_k' \\ & +\frac{\partial H}{\partial \varphi_2}\sum_{k=1}^{n}\frac{\partial \varphi_2}{\partial q_k}q_k'. \end{aligned}\right\} \tag{18}$$

For given F_1, F_2 and φ_1, φ_2 functions, eq. (18) is one of the orbit equations. For given F_1, F_2 and $q_k=q_k(q_1)$ functions, eq. (18) is a partial differential equation for φ_1 and φ_2. For given F_1 and F_2, eq. (18) is an orbit equation and can be used to select φ_1 and φ_2 to facilitate obtaining the solution.

The other orbit equations corresponding to eq. (9) become for $i=3(1)n$:

$$F_i+\varphi_i=q_i'(F_1+\varphi_1)+q_i''H \tag{19}$$

i.e. every orbit equation introduces one new φ_i function.

The previously mentioned set of eqs. (10) in a modified form becomes:

$$\left.\begin{aligned} \ddot{x}&=F_1(x, y, \dot{x}, \dot{y})+\varphi_1(x, y) \\ \ddot{y}&=F_2(x, y, \dot{x}, \dot{y})+\varphi_2(x, y). \end{aligned}\right\} \tag{20}$$

Eq. (16) for this case is

$$F_2+\varphi_2=y'(F_1+\varphi_1)+y''(\dot{x})^2 \tag{21}$$

from which $\dot{x}^2$ can be computed, giving

$$\dot{x}^2=H(x, y, y', y'', \varphi_1, \varphi_2). \tag{22}$$

The equation of the orbit, corresponding to eq. (18) is

$$\left.\begin{aligned} 2F_1+2\varphi_1= & \frac{\partial H}{\partial x}+y'\frac{\partial H}{\partial y}+y''\frac{\partial H}{\partial y'}+y'''\frac{\partial H}{\partial y''} \\ & +\left(\frac{\partial \varphi_1}{\partial x}+y'\frac{\partial \varphi_1}{\partial y}\right)\frac{\partial H}{\partial \varphi_1}+\left(\frac{\partial \varphi_2}{\partial x}+y'\frac{\partial \varphi_2}{\partial y}\right)\frac{\partial H}{\partial \varphi_2}. \end{aligned}\right\} \tag{23}$$

This is an ordinary third order differential equation for $y(x)$, i.e. for the function representing the orbit, if F_1, F_2, φ_1 and φ_2 are given. It also can be considered as a partial differential equation for $\varphi_1(x, y)$ and $\varphi_2(x, y)$, i.e. for the modifying functions, if F_1, F_2 and $y = y(x)$ are given. Finally, eq. (23) can also serve as a guide in selecting the φ_1, and φ_2 functions for given F_1 and F_2 so that the solution can be represented by functions of known geometric properties.

As summary of this chapter, the following remarks are offered:

(i) For an n degree of freedom system there are n modifying functions: $\varphi_1 \dots \varphi_n$ which depend on the n generalized position coordinates, $q_1 \dots q_n$. After elimination of the time the $2n$-th order system is reduced to a system of $(2n-1)$-th order, represented by $n-1$ equations. The first equation is of the 3-rd order. It includes φ_1, φ_2 and their partial derivatives, the latter ones linearly. The remaining $(n-2)$ equations each include φ_1, φ_2 and only one of the φ_k functions $(k = 3(1)n)$.

(ii) If the F_k functions and the $q_k(q_1)$ functions are given the n modifying functions φ_k will have to be determined from $n-1$ equations, one of these being a partial differential equation, i.e. the φ_k functions are not uniquely determined.

(iii) For given F_k functions (properly) selected φ_k functions will completely determine the solution.

3. Applications

3.1. General considerations

It will be shown that the above conclusion marked (ii) allows the formulation of generalized guidance equations on one hand (section 3.1.1) and offers a new presentation of the Encke method on the other hand (section 3.1.2). Conclusion (iii) points the way to the establishment of new approximate solutions (section 3.1.3).

3.1.1. *Generalized guidance approach*

The combination of eqs. (1) and (15) describes a dynamical system where the "forcing functions" are split into two parts:

$$\ddot{q}_i = F_i(q_k, \dot{q}_k) + \varphi_i(q_k). \tag{24}$$

The first part (F_i) is now considered as the given force field, the second part (φ_i) as the guidance force along an orbit. To determine the n guidance force components (φ_i), eqs. (18) and (19) are used. Since there are only $(n-1)$

equations, no unique solution is available, in fact, the guidance forces $\varphi_3, \varphi_4 \ldots \varphi_n$ will be expressed as functions of φ_1 and φ_2. It is noted that φ_1 and φ_2 are not independent but related by a partial differential equation. From the point of view of guidance this is one of the advantages of the present approach since a great variety of guidance functions result from the analysis. In actual cases, the first guidance function might be determined as a function of the position coordinates and of the second guidance function (φ_2). Even if φ_2 is fixed, the first guidance function is not determined uniquely since it appears as a solution of a partial differential equation. This equation has the general form:

$$L\left(q_1, \varphi_1, \varphi_2, \frac{\partial\varphi_1}{\partial q_k}, \frac{\partial\varphi_2}{\partial q_k}\right) = 0 \tag{25}$$

where L is a linear function of the partial derivatives.

The determination of the set of guidance functions therefore is reduced to first finding the general solution of eq. (25), for instance, in the form o $\varphi_1 = \varphi_1(\varphi_2, q_1)$. Using then eqs. (19) the remaining guidance functions are determined as

$$\left.\begin{aligned} \varphi_i &= \varphi_i(\varphi_2, q_1) \\ i &= 3(1)n. \end{aligned}\right\} \tag{26}$$

Optimizing conditions imposed on the problem might require finding the minimum of the total force vector, (Ψ_1) of work (Ψ_2) or of power (Ψ_3), i.e.:

$$\Psi_1 = \sqrt{\varphi_1^2 + \varphi_2^2 + \ldots \varphi_n^2} \quad \text{or} \quad \Psi_2 = \int \sum_{k=1}^{n} \varphi_k \mathrm{d}q_k \quad \text{or} \quad \Psi_3 = \int \sum_{k=1}^{n} \varphi_k \mathrm{d}\dot{q}_k. \tag{27}$$

3.1.2. *Generalized Encke method*

The Encke perturbation method computes the difference between the actual and a fictitious precomputed ("nominal") trajectory. The coordinates of this nominal trajectory can be represented by closed form solutions of a differential equation. The method is based on the premise that if the differential equation for the nominal trajectory is a "slight" modification of the ifferential equation associated with the original problem, then the deviation between the actual and the nominal trajectories is small.

The Encke method [3] is summarized here for comparison using the notations of this paper and properly selected units:

$$\text{let} \qquad \ddot{q}_i = -\frac{q_i}{r^3} + P_i \tag{28}$$

where q_i is the i-th position coordinate, $r^2 = \sum_{i=1}^{3} q_i^2$ and P_i is the i-th component of the perturbation. These equations represent the actual trajectory

while the reference trajectory satisfies the following differential equation

$$\ddot{p}_i = -\frac{p_i}{r_0^3} \tag{29}$$

where p_i is the i-th coordinate of the two body reference orbit and

$$r_0^2 = \sum_{i=1}^{3} p_i^2.$$

Introducing for the difference between the actual and the reference trajectories π_i one obtains the differential equation, which the Encke method integrates:

$$\left.\begin{aligned} \ddot{\pi}_i &= \frac{1}{r_0^3}\left[q_i\left(1-\frac{r_0^3}{r^3}\right)-\pi_i\right]+P_i \\ \pi_i &= q_i - p_i. \end{aligned}\right\} \tag{30}$$

For actual computational purposes eq. (30) is written as

$$\ddot{\pi}_i = \frac{a}{r_0^3} q_i - \frac{1}{r_0^3}\pi_i + P_i \tag{31}$$

where the computation of

$$a = 1 - \frac{p_0^3}{[\sum(\pi_i + p_i)^2]^{3/2}}$$

is facilitated by tables.

If the reference orbit is "close" to the actual, large integration steps can be taken, therefore, the efficiency of the Encke method depends on the proper choice of the reference orbit. This is not difficult when planetary motions are studied, i.e. when frequent rectifications (selection of new reference orbits) are not necessary. For purposes of lunar trajectories, or in general, for three body orbit calculations the original Encke scheme has limited significance especially in the regions where the deviations from two body orbits are large.

It seems to be a logical extension of the Encke method to investigate reference orbits which are not based on two body calculations but which approximate the trajectories associated with the three body fields. Let

$$\ddot{q}_i = f_i(q_k, \dot{q}_k) \tag{32}$$

be the i-th differential equation describing the actual motion of a body in a given (f_i) force field and let the reference orbit be described by

$$\ddot{p}_i = f_i(p_k, \dot{p}_k) + \varphi_i(p_k) \tag{33}$$

where p_i is the i-th coordinate of the reference orbit.

The partial differential equations for (φ_i) which are associated with eq. (33) are obtained from eqs. (18) and (19) by writing p_i for q_i and f_i for F_i:

$$2f_1 + 2\varphi_1 = \chi_1\left(p_k, p_k', p_k'', p_2''', \varphi_1, \varphi_2, \frac{\partial\varphi_1}{\partial p_k}, \frac{\partial\varphi_2}{\partial p_k}\right) \tag{34}$$

and

$$f_i + \varphi_i = p_i'(f_1 + \varphi_1) + p_i'' H(p_k, p_k', p_2'', \varphi_1, \varphi_2), \quad \text{for} \quad i = 3(1)n. \tag{35}$$

A set of simple $p_i(p_1)$ functions can be constructed which represent approximately the solution. Substituting these functions, eqs. (34) and (35) will give several sets of φ_i functions. These sets are subjected to a minimization process and a single set of φ_i functions is obtained. This set of φ_i with the associated $p_i(p_1)$ functions complete the construction of the reference orbit

Introducing now the deviations of the actual orbit from the reference orbit by

$$\pi_i = q_i - p_i$$

and subtracting eq. (33) from (32) one obtains:

$$\ddot{\pi}_i = f_i(q_k, \dot{q}_k) - f_i(p_k, \dot{p}_k) - \varphi_i(p_k) \tag{36}$$

where $\pi_i(t)$ is to be determined, $q_k(t) = \pi_k(t) + p_k(t)$ and $p_k(t)$ is known.

This concludes the generalization of the Encke method excepting the important fact that Encke writes the final eq. (31) in a form specifically suited for numerical work. The corresponding step in this generalized treatment can be accomplished only if the specific f_i functions describing the field of an actual problem are given.

3.1.3. *Approximate solutions*

The problem of approximate solutions is twofold: firstly, a method of constructing such solutions is to be established and secondly, the accuracy is to be estimated. It will be assumed that the solution of the original system of differential equations is not available, so the estimation of the error of the approximate solution must be more sophisticated than comparing two functions in a given domain. Error estimates are intimately associated with the functions (φ_i) previously called guidance or modifying functions as was shown before.

In section 3.1.1 the guidance problem was discussed and it was shown how the φ_i functions are determined along a prescribed trajectory, i.e. how the given force field is modified along a given orbit. The present chapter deals with the modification of the entire force field, so that approximate orbits subject to arbitrary initial conditions can be represented in "closed" form.

The Encke method is a special perturbation technique and the aim of the previous section (3.1.2) was to generalize its reference orbit aspects. The purpose of the present chapter is to show how the inverse trajectory concept can be applied to obtain approximate general solutions and how to estimate the accuracy of these solutions.

Eqs. (18) and (19) with $\varphi_i \equiv 0$ represent the orbit in a given F_i force field and it is assumed that the $q_i = q_i(q_1)$ functions which satisfy these equations with given initial conditions are not available. In order to find an approximate solution to the $\varphi_i \equiv 0$ actual problem, one finds the $\varphi_i \neq 0$ sets for which the general solution of the orbit differential equations can be given. In other words, the exact general solution for the orbit in a modified field $F_i + \varphi_i$ is determined first. The approximate general solution of the actual problem is represented, therefore, by the exact general solution of the approximate problem.

Since the difference between the solutions of the actual and modified differential equations are in general proportional to the modifying functions applied, the motion in the original force field will differ from the motion in the modified field by an amount which is proportional to the field distortion. The φ_i functions represent the distortion of the field and the smaller these distortions are, the better the approximation becomes.

The description of the analysis involved is simple. Referring to eqs. (18) and (19) one selects an arbitrary set of $\varphi_i(q_k)$ functions for which the $q_i = q_i(q_1)$ solutions can be represented by "simple" functions. This does not require the precise specification of the φ_i set, only the general functional forms of its members. The next step is to minimize the values of the members of the φ_i field distortion set in the domain of interest. The result will be a set o φ_i functions which will allow a simple representation of the general solution (q_i) and at the same time will minimize the deviation from the actual solution.

Three remarks are of some importance:

(i) The error estimate for the approximate general solution is associated with a domain in which the solution is applicable.

(ii) The error estimate in general will be very crude unless the special form of the differential equations is taken into account.

(iii) The field distortion functions might introduce large deviations if the domain is "large". Approximate solutions consisting of several functions which are matched at the boundaries of their respective domain might have to be considered.

3.2. The planar restricted three body problem

3.2.1. *Reduction to the second order*

The differential equations of motion of the title problem in a rotating Cartesian rectangular coordinate system with origin at the mass center of the two principal bodies (see fig. 1) can be found in any standard reference [4] in the following form:

$$\left.\begin{aligned} \ddot{x} &= n^2x + 2n\dot{y} - \frac{\mu(x+a)}{r^3} - \frac{\nu(x-b)}{\varrho^3} \\ \ddot{y} &= n^2y - 2n\dot{x} - \frac{\mu y}{r^3} - \frac{\nu y}{\varrho^3} \end{aligned}\right\} \tag{37}$$

where n is the angular velocity of the system, r is the distance between m_1 and m, ϱ is the distance between m_2 and m, $\mu = k^2m_1$, $\nu = k^2m_2$

$$a = \frac{lm_2}{m_1+m_2} \quad \text{and} \quad b = \frac{lm_1}{m_1+m_2}. \tag{38}$$

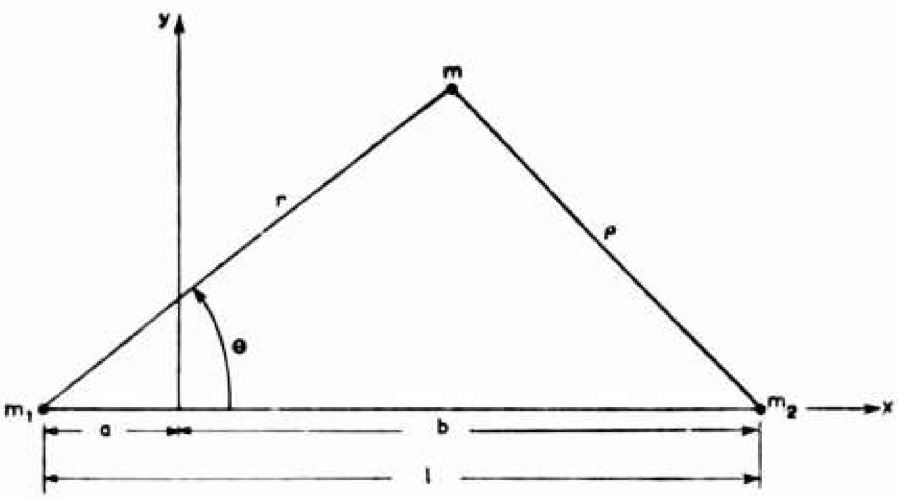

Fig. 1. Coordinate system for the restricted 3-body problem.

The Jacobi integral is

$$(\dot{x})^2 + (\dot{y})^2 - n^2(x^2+y^2) - \frac{2\mu}{r} - \frac{2\nu}{\varrho} = 2C. \tag{39}$$

The third order differential equation of the orbit, containing x, y, y', y'' and y''' is obtained by using eq. (7). The order might be reduced to two if use is made of the Jacobi integ al. These steps will be executed below, but first a transformation to polar coordinates will be made, by the relations:

$$x + a = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{40}$$

After substitution, the equations of motion (37) become:

$$\left.\begin{aligned}\ddot{r} &= r(\dot{\theta}+n)^2 - an^2\cos\theta - \frac{\mu}{r^2} - \frac{\nu(r-l\cos\theta)}{\varrho^3}\\ \ddot{\theta} &= -2\frac{\dot{r}}{r}(\dot{\theta}+n) + \frac{an^2\sin\theta}{r} - \frac{\nu l\sin\theta}{r\varrho^3}\end{aligned}\right\} \tag{41}$$

and the Jacobi integral assumes the form:

$$(\dot{r})^2 + (r\dot{\theta})^2 - n^2r^2 + 2ran^2\cos\theta - \frac{2\mu}{r} - \frac{2\nu}{\varrho} = 2C \tag{42}$$

where

$$\varrho^2 = r^2 - 2rl\cos\theta + l^2. \tag{43}$$

It is noted that using

$$V = -\frac{\mu}{r} - \frac{\nu}{\varrho} + ran^2\cos\theta \tag{44}$$

eqs. (41) and (42) can be written as

$$\left.\begin{aligned}\ddot{r} &= r(\dot{\theta}+n)^2 - \frac{\partial V}{\partial r}\\ r\ddot{\theta} &= -2\dot{r}(\dot{\theta}+n) - \frac{1}{r}\frac{\partial V}{\partial\theta}\end{aligned}\right\} \tag{45}$$

and

$$(\dot{r})^2 + (r\dot{\theta})^2 - n^2r^2 + 2V = 2C. \tag{46}$$

It is noted that the (r, θ) polar coordinate system is originated at m_1 and not at the origin of the (x, y) coordinate system and therefore direct comparison of eqs. (45) and (46) with reference [4] might be misleading. The third order differential equation, corresponding to eq. (7) is obtained if the following substitutions are made:

$$q_1 = \theta, \quad q_2 = r, \quad f_1 = -\frac{2\dot{r}(\dot{\theta}+n)}{r} - \frac{1}{r^2}\frac{\partial V}{\partial\theta}, \quad f_2 = r(\dot{\theta}+n)^2 - \frac{\partial V}{\partial r}.$$

In order to arrive at a second order differential equation, use will be made also of the Jacobi integral. Writing $\dot{r} = r'\dot{\theta}$ where $r' = (\mathrm{d}r)/(\mathrm{d}\theta)$, $(\dot{\theta})^2$ can be expressed using the Jacobi integral (46):

$$(\dot{\theta})^2 = \frac{2(C-V) + n^2r^2}{r^2 + (r')^2} = \frac{g^2(r, \theta)}{r^2 + (r')^2} \tag{47}$$

where the $g(r, \theta)$ function was introduced for simplicity's sake and its physical meaning is the magnitude of the velocity of the mass m in the rotating

coordinate system. It is noted that $\dot{\theta}$ has not been expressed from an equation corresponding to (4) in the general treatment since the Jacobi integral could be used instead.

Corresponding to eq. (3) we have

$$\ddot{r} = r'\ddot{\theta} + r''(\dot{\theta})^2 \tag{48}$$

or according to (45)

$$r(\dot{\theta}+n)^2 - \frac{\partial V}{\partial r} = -\frac{r'}{r}\left[2\dot{r}(\dot{\theta}+n) + \frac{1}{r}\frac{\partial V}{\partial \theta}\right] + r''(\dot{\theta})^2. \tag{49}$$

This equation corresponds to eq. (4). If the Jacobi integral would not exist, one would solve eq. (49) for $(\dot{\theta})^2$ and would proceed as in the general case.

Using the expression for $(\dot{\theta})^2$ as given by eq. (47), observing that

$$\frac{\partial V}{\partial r} = n^2 r - \frac{1}{2}\frac{\partial g^2}{\partial r}$$

$$\frac{\partial V}{\partial \theta} = -\frac{1}{2}\frac{\partial g^2}{\partial \theta}$$

$$\dot{r} = r'\frac{g(r,\theta)}{\sqrt{r^2+(r')^2}},$$

and substituting in eq. (49) the result is:

$$g\left(r'' - 2\frac{(r')^2}{r} - r\right) + [r^2+(r')^2]\left(\frac{\partial g}{\partial \theta}\frac{r'}{r^2} - \frac{\partial g}{\partial r}\right) = \frac{2n}{r}[r^2+(r')^2]^{3/2}. \tag{50}$$

This second order differential equation for the $r = r(\theta)$ function represents the orbit associated with the planar restricted three body problem.

If the reciprocal transformation for the radius vector is introduced, eq. (50) assumes the following form:

$$g(u''+u) = \left[\left(\frac{u'}{u}\right)^2 + 1\right](u^2 g_u - u' g_\theta) - 2n\left[\left(\frac{u'}{u}\right)^2 + 1\right]^{\frac{3}{2}} \tag{51}$$

where $u = 1/r$, the subscripts represent partial derivatives, and

$$g^2 = 2C + \frac{n^2}{u^2} - 2\frac{an^2}{u}\cos\theta + 2u\,[\mu + \nu(1 - 2lu\cos\theta + l^2u^2)^{-1/2}]. \tag{52}$$

Eq. (51) is the result of the derivation. Its general solution would represent the solution of the restricted three body problem.

Two remarks related to eq. (51) might be in order:

(i) Due to the choice of the coordinate system and because of the structure of the differential equation, the latter reduces easily to the corresponding two body problem by writing

$$\nu = 0 \quad \text{and} \quad n = 0.$$

For this case, which corresponds to the motion of m in the field of m_1 in a fixed coordinate system, eq. (52) becomes:

$$g^2 = 2\mu u + 2C \quad \text{and so} \quad g_u = \mu/g. \tag{53}$$

Eq. (51) gives:

$$u'' + u = [(u')^2 + u^2]\,\mu/g^2 \tag{54}$$

or

$$u + u'' = \mu/h^2 \tag{55}$$

where h is the constant of integration of the momentum, i.e.

$$h = \dot{\theta} r^2 = \dot{\theta}/u^2.$$

It is noted that eq. (47), using the (u, θ) variables can be written as

$$(\dot{\theta})^2 = \frac{g^2 u^4}{u^2 + (u')^2} \tag{56}$$

and therefore

$$\frac{(u')^2 + u^2}{g^2} = \left(\frac{u^2}{\dot{\theta}}\right)^2 = \frac{1}{h^2}$$

which relation was used to obtain eq. (55).

The fact that eq. (54) is identical with the well known form of the two body equation, i.e.

$$u + u'' = \text{constant}$$

can also be shown without resorting to the momentum integral as follows:

Eq. (54) can be written as

$$\mu[(u')^2 + u^2] = 2(C + \mu u)(u + u'')$$

which by differentiation becomes

$$\mu[(u')^2 + u^2]' = 2\mu u'(u + u'') + 2(C + \mu u)(u + u'')'. \tag{57}$$

Comparing (57) with the identity:

$$[(u')^2 + u^2]' = 2u'(u + u'') \tag{58}$$

we have $u + u'' = \text{constant}$, provided that the velocity is not zero.

(ii) The Lagrangian solutions follow rather elegantly from eq. (51) if $u = u_0 =$ constant solutions are searched for. Since the $g(u, \theta)$ function is identical with the velocity magnitude in the rotating system and since the Lagrangian solutions are stationary in this system,

$$g \equiv 0.$$

This reduces the differential eq. (51) to

$$\left(\frac{\partial V}{\partial u}\right)_{u_0} + \frac{n^2}{u_0^3} = 0 \tag{59}$$

or

$$\mu u_0^3 + \frac{\nu}{\varrho_0^3}(1 - u_0 l \cos\theta_0) = n^2(1 - a u_0 \cos\theta_0). \tag{60}$$

Eliminating a and n^2 by

$$a = \frac{\nu l}{\mu + \nu} \quad \text{and} \quad n^2 = \frac{\mu + \nu}{l^3} \tag{61}$$

and introducing the $\sigma = \mu/\nu$ and $x = u_0 l$ notation, the equation for the libration points (60) becomes:

$$\sigma(x^3 - 1) + (x\cos\theta_0 - 1)\left[1 - \frac{x^3}{(x^2 + 1 - 2x\cos\theta_0)^{3/2}}\right] = 0. \tag{62}$$

3.2.2. *Guidance*

It was shown in section 3.1.1 that the number of guidance functions φ_i equals the number of degrees of freedom. Since the planar restricted three body problem has two degrees of freedom, φ_1 and φ_2 are to be introduced and evaluated. Eqs. (45) in their modified forms are:

$$\left.\begin{aligned} \ddot{r} &= r(\dot\theta + n)^2 - \frac{\partial V}{\partial r} + \varphi_1(r, \theta) \\ r\ddot{\theta} &= -2\dot{r}(\dot\theta + n) - \frac{1}{r}\frac{\partial V}{\partial \theta} + \varphi_2(r, \theta). \end{aligned}\right\} \tag{63}$$

Reduction of this problem to a second order differential equation is possible only if the Jacobi integral is applicable, that is if φ_1 and φ_2 are components of a conservative field, or if

$$\frac{\partial \varphi_1}{\partial \theta} = \frac{\partial \varphi_2 r}{\partial r}.$$

In other words if the guidance functions can be derived from a potential, Φ, then the potential function (V) of the original problem can be modified

and the equations of motion become

$$\left.\begin{aligned} \ddot{r} &= r(\dot{\theta}+n)^2 - \frac{\partial V^*}{\partial r} \\ r\ddot{\theta} &= -2\dot{r}(\dot{\theta}+n) - \frac{1}{r}\frac{\partial V^*}{\partial \theta} \end{aligned}\right\} \tag{64}$$

where

$$V^* = V + \Phi(r, \theta). \tag{65}$$

This way, the fourth order system can be reduced to a second order differential equation, containing r, r', r'', θ, Φ, Φ_r, Φ_θ. It is noted that by introducing Φ instead of using independent φ_1 and φ_2 functions, the problem is restricted to conservative guidance forces, but this restriction is not necessary and it is made for convenience's sake only. The radial and tangential components of the guidance force are determined, once the guidance potential (Φ) is found, by

$$\left.\begin{aligned} \varphi_1 &= -\frac{\partial \Phi}{\partial r} \\ \varphi_2 &= -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}. \end{aligned}\right\} \tag{66}$$

The second order differential equation of the orbit can be established simply by substituting

$$(g^*)^2 = 2(C - V^*) + \frac{n^2}{u^2} \tag{67}$$

and

$$V^* = V + \Phi$$

in eq. (51). Since $V(u, \theta)$ is given (44) and

$$\frac{\partial g^*}{\partial u} = -\frac{1}{g^*}\left(V_u + \Phi_u + \frac{n^2}{u^3}\right),$$

$$\frac{\partial g^*}{\partial \theta} = -\frac{1}{g^*}(V_\theta + \Phi_\theta),$$

the basic guidance equation from (51) becomes

$$\left.\begin{aligned} & g^*\left\{2n\left[\left(\frac{u'}{u}\right)^2 + 1\right]^{\frac{3}{2}} + g^*(u + u'')\right\} \\ &\qquad = \left[\left(\frac{u'}{u}\right)^2 + 1\right]\left[u'(V_\theta + \Phi_\theta) - u^2\left(V_u + \Phi_u + \frac{n^2}{u^3}\right)\right]. \end{aligned}\right\} \tag{68}$$

This is the equation which corresponds to eq. (18) or (19). The differential eq. (68) is of the second order since the modifying function was derived from a potential. For given $u = u(\theta)$ this equation is a partial differential equation for the guidance potential (Φ). The erroneous observation might be made that partial derivatives would not appear (see eq. (19)) if a potential function for the guidance forces would not have been introduced. On the other hand it should be observed that eq. (18) which did not use a potential function for the modifying functions, does contain partial derivatives and so does eq. (25). Therefore, we conclude that the guidance equation is always a partial differential equation either for the guidance functions or for the guidance potential.

At this point, to illustrate the method, we select an orbit and show how the guidance potential is determined. Consider the

$$u = \alpha + \beta \cos \theta \tag{69}$$

conic sections and require that the mass m describe this orbit.

It can be shown that conic sections do not satisfy the differential equations of the restricted three body problem [5], therefore, selecting these orbits, triviality ($\Phi = 0$) is avoided. The practical significance of such orbits is also demonstrated in [5].

Substituting eq. (69) into (68) the partial differential equation for Φ becomes:

$$L(\Phi, \theta) = M(\theta) \frac{\partial \Phi}{\partial \theta} + \frac{\partial \Phi}{\partial u}. \tag{70}$$

From this equation $\Phi(u, \theta)$ is determined keeping in mind that the investigation refers to a specific type of orbits, i.e. to those which are described by eq. (69). By selecting any other family of curves for the path and substituting these into eq. (68), the resulting equation will have the same general form as eq. (70) and only the $L(\Phi, \theta)$ and $M(\theta)$ functions will change. It is to be noted that the dependence of L on Φ is always of the same form, i.e.

$$L(\Phi, \theta) = P(\theta) \sqrt{Q(\theta) - 2\Phi} + \Phi R(\theta) + S(\theta) \tag{71}$$

and for different paths only the P, Q, R, S functions will be different.

The partial differential eq. (70) is solved by the standard method, i.e.

$$\frac{\mathrm{d}\theta}{M(\theta)} = \mathrm{d}u = \frac{\mathrm{d}\Phi}{L(\Phi, \theta)} \tag{72}$$

from which

$$u - \int \frac{\mathrm{d}\theta}{M(\theta)} = \Psi_1 \tag{73}$$

and

$$\Phi - \Omega(\theta) = \Psi_2 \tag{73}$$

where $\Phi = \Omega(\theta)$ is the solution of

$$\frac{d\Phi}{d\theta} = \frac{L(\theta, \Phi)}{M(\theta)}. \tag{74}$$

The general solution of the partial differential eq. (70) is

$$\Psi_2 = \Psi(\Psi_1)$$

or

$$\Phi = \Omega(\theta) + \Psi\left(u - \int \frac{d\theta}{M(\theta)}\right), \tag{75}$$

where Ψ is an arbitrary function of its argument.

The determination of the actual guidance force components is made by means of eqs. (66):

$$\varphi_1 = -u^2 \frac{\partial\Phi}{\partial u} = -u^2 \Psi_u$$

and

$$\varphi_2 = -u \frac{\partial\Phi}{\partial\theta} = -u(\Omega' + \Psi_\theta). \tag{76}$$

For the conic section orbits the M, P, Q, R, S functions are as follows:

$$M(\theta) = \frac{\beta \sin\theta}{(\alpha + \beta\cos\theta)^2} \tag{77}$$

$$P(\theta) = -2n \frac{\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta\cos\theta}}{(\alpha + \beta\cos\theta)^3} \tag{78}$$

$Q(\theta) = g^2(\theta)$ as shown in eq. (52),

$$R(\theta) = \frac{2\alpha}{\alpha^2 + \beta^2 + 2\alpha\beta\cos\theta} \tag{79}$$

and

$$S(\theta) = -(M + \tfrac{1}{2}QR). \tag{80}$$

The characteristic eqs. (73) furnish the solutions:

$$\Psi_1 = u - \beta\cos\theta - 2\alpha \ln\sin\theta - \frac{\alpha^2 + \beta^2}{\beta} \ln\tan\frac{\theta}{2} \tag{81}$$

and

$$\Psi_2 = \Phi - \Omega(\theta)$$

where $\Omega(\theta)$ is obtained from an equation corresponding to eq. (74):

$$\frac{d\Phi}{d\theta} = \frac{P}{M}\sqrt{Q-2\Phi} + \Phi\frac{R}{M} + \frac{S}{M}. \tag{82}$$

This equation can be reduced to an Abel type differential equation and this way the guidance potential according to eq. (75) is determined, subject to the selection of an arbitrary Ψ function.

3.2.3. *Encke method*

Three body trajectories which connect the neighborhood of m_1 with the neighborhood of m_2 are of considerable practical interest. Such orbits often are constructed of three parts. The first part is a two body approximation fitted to that part of the trajectory which is in the vicinity of m_1. The third part is another two body approximation fitted in the neighborhood of m_2. The second, in-between part of the trajectory, the matching of these two body fits and the problem of approximating the trajectory in the region where the force field is essentially a three body field, is seldom treated. Since the classical Encke method or the variation of parameters method are both ideally suited for treating domains where two body fields dominate, in this chapter, we will concentrate on the "essentially" three body regime.

Trajectories which in a rotating coordinate system show up as figure 8 or S-shaped curves, contain inflection points and in these regions straight line approximations are of interest. Neither the classical variation of parameters nor the conventional Encke method is designed to entertain these regions.

As a reference orbit the

$$v = \alpha \sin\theta + \beta\cos\theta \tag{83}$$

straight line is used, with $l > 1/\beta \gg |1/\alpha|$, $\alpha < 0$ corresponding to the actual problem, i.e. the line will intersect the x-axis (fig. 1) between the origin and m_2 with a small angle of positive inclination. Since the variable u is reserved for the actual orbit, v is used for the reference orbit (compare with eqs. (32) and (33)). Just as the Encke method maintains the time as the common independent variable for both the actual and the reference orbit, in the present technique the independent angular variable will be considered common to both orbits.

Substituting the reference orbit into eq. (68), one finds that (since $v + v'' = 0$) the governing equation for the reference orbit becomes:

$$2n\sqrt{\alpha^2+\beta^2} = (\alpha\sin\theta + \beta\cos\theta)^3 g_v{}^* - [\alpha\beta\cos 2\theta + \tfrac{1}{2}(\alpha^2-\beta^2)\sin 2\theta]\, g_\theta{}^*. \tag{84}$$

The general solution is easily obtained by the method of characteristics:

$$g^* = \Psi(v + P) - \frac{2n}{(\alpha^2 + \beta^2)^{1/2}} \ln \tan \left(\theta + \operatorname{arc} \tan \frac{2\alpha\beta}{\alpha^2 - \beta^2}\right) \tag{85}$$

where Ψ is an arbitrary function and

$$P = \sqrt{\alpha^2 + \beta^2} \ \ln \ \frac{\sqrt{\alpha^2 + \beta^2} - v}{2(\beta \sin\theta - \alpha \cos\theta)}. \tag{86}$$

The general form of the φ_1 and φ_2 functions require now the determination of Φ_v and Φ_θ. According to eqs. (76):

$$\left.\begin{aligned} \varphi_1 &= -v^2\Phi_v = v^2(g^* g_v^* + V_v) + \frac{n^2}{v} \\ \varphi_2 &= -v\Phi_\theta = v(g^* g_\theta^* + V_\theta). \end{aligned}\right\} \tag{87}$$

This completes the problem, since g^* is given by (85) and V by (44).

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